

Hilbert's Program

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1 Introduction

After the discovery of the set-theoretic paradoxes at the turn of the 20th Century, physicist and mathematician Hermann Weyl called classical mathematics a house “to a large degree built on sand” (Weyl (1918), 1). Hilbert's Program (HP) emerged as the leading response to this skepticism about the power of classical reasoning and, arriving on the mathematical scene in the early 1900s, HP was intended to be a one-two punch. On the one hand, Hilbert and his School thought that if it were possible to clean up the classical concepts by formalizing them in a deductive theory and then proving, mathematically, that the theory could not produce a contradiction (punch one), then such a proof would constitute epistemic justification for believing that theory (punch two). Because if a theory is consistent, then it follows that there's a model of it in which all of its axioms are true. So, if we believe the axioms, the axioms are true, and the consistency proof justifies our belief that the axioms are true, then it seems to follow that we have knowledge of the axioms, and that such mathematical knowledge is consistent. HP was on the road to victory. Alas, it was not to be.

In the 1930s a young mathematician and researcher in the Hilbert School named Kurt Gödel announced his first and second incompleteness theorems. The effect was immediate. Most working mathematicians thought that if the first incompleteness theorem (G1) didn't portend HP's death, then Gödel's second theorem (G2) certainly did. G1 says, roughly, that in any formal mathematical theory that can produce a sufficient amount of arithmetical theorems, it is always possible to produce a true but unprovable sentence

in that theory.¹ In short, what is called a theory’s “Gödel Sentence” is a sentence that we must assume to be true without proof. G2 goes further and says that, again in a sufficiently strong formal theory, it is possible to code the Gödel Sentence so that it says that the theory itself is consistent. Hence, while the first theorem produces the existence of the troubling true but unprovable sentence in the theory, it’s the second that seems to do the real damage to HP. For if we code that sentence up in the theory to say that that the theory itself is consistent, then it seems to follow from the first theorem that said sentence is true but unprovable. In short, Gödel’s first and second theorems taken together seem to state a simple and, to HP, devastating fact: the consistency of a mathematical theory is unprovable by that theory. It’s precisely that fundamental fact that seems to dash all hopes and the history of metamathematics since the late 1930s seems to bear this out. Current interest in metamathematics centers primarily around competing metamathematical disciplines that gained steam after Gödel’s result — the dominance of set theory and category theory in metamathematics are outstanding examples. On the other hand, the direction of classical proof theory turned — Kreisel’s unwinding program and Feferman’s conservativity results famously — and any inkling of rehabilitating, mining, or reworking HP consistency program along its original search for a consistency proof has for the most part vanished.

In the entry that follows we’ll look at HP in two parts. In the first part, we’ll look at the historical context and background to HP. This will set up the motivation for and understanding of some of the details surrounding HP and how Hilbert himself understood it. In the second part of the entry, we will discuss the emergence of HP in the 1920s with the Hilbert School in Göttingen and its various offshoot programs. This will give some more detail to what and how the mature version of HP was to be carried out.

¹ Below we will return to what it means for a formal theory to “say” something.

2 Origins of HP

In late 1899, Hilbert announced the formation of what he took to be a novel approach to the foundations of mathematics, his axiomatic method. Starting around 1891, Hilbert began exploring the approach to systematically treating the informal reasoning processes and patterns implicit in the practice of mathematical theorem proving. It is both a *logical* method of investigation that aims to explicate the techniques of reasoning and the logical and mathematical relations implicit in an informal mathematical theory, and equally important to Hilbert, the axiomatic method is an *epistemological* method of investigation, the aim of which is to explicate the premisses and proof methods—the necessary and sufficient conditions—called upon in the justification of a mathematical assertion. In explicating the conditions by which a mathematical assertion is provable, Hilbert’s axiomatic method aims to show that the axioms and inference rules are epistemically justified. Hilbert’s most mature deployment of the method during his “geometric period” is his *Grundlagen der Geometrie*, written as a *Festschrift* for the unveiling of the Gauss-Weber monument in Königsberg.² In the introduction, Hilbert announces that his “novel approach” consists in finding a “*simple* [*einfaches*] and *complete* [*vollständiges*] system of mutually *independent* [*unabhängiger*] axioms,” that illuminate the meaning and significance of the most important geometric assertions ((1899), 1).

In chapter two of (1899) Hilbert demonstrates that his system (HG) is consistent [*Widerspruchslös*] relative to the consistency of the arithmetic of the reals. In his debate with Frege, Hilbert writes that if the “arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist” (Frege (1980), 39). Philosophical accounts of Hilbert’s philosophy of mathematics stress the consistency criterion and take it to legitimate an inference from consistency to truth, and from truth to the existence of the points, lines, and planes over which HG’s quantifiers range. But it might come as a surprise to learn

² In what follows, we refer to the *Grundlagen der Geometrie* as (1899); likewise by date for the other works of the geometric period, all of which are found in Hallett and Majer (2004); translations from Hallett and Majer (2004) are my own.

that consistency is *not* listed in the introduction to (1899) as a goal of the axiomatic method applied to geometry, and the majority of chapter two is devoted to independence proofs for the five axiom groups.³ What explains the incongruity between the narrow focus on consistency by contemporary readers, and the much wider range of conditions that Hilbert states? One of the reasons is the influence of Hilbert’s later consistency program. But a deeper reason is that of the following four conditions—simplicity, completeness, consistency, and independence—that Hilbert lists, it is assumed that the only condition that admits of precise treatment *and* carries epistemic significance is the consistency condition.

But Hilbert claims that if all four conditions are satisfied, if a set of axioms is “compatible [*Verträglich*],” roughly meaning that the set “coheres,” then the mathematician is permitted to believe that the informal mathematical theory presented by the set of axioms is epistemically justified. In the following section we show that implicit in this concept of coherence are the particular metamathematical ingredients for at least three important concepts of the later proof-theoretic HP, and, through an analysis of Hilbert’s proof, in chapter five of (1899), of Desargues’ Theorem we show that nascent in his use of points and lines at infinity is the concept of a *conservative extension*. Through an analysis of some of the central metamathematical properties that emerge from Hilbert’s early geometric program, our goal is to suggest that HP is better understood in terms of its emergence from Hilbert’s geometric program than in light of the set-theoretic paradoxes.

2.1 Extension, Preservation, and Consistency

Consistency and independence are complements of one another. If a proposition ϕ and its negation are unprovable from (and so, independent of) a set of propositions Φ , then Φ is consistent with ϕ and consistent with $\neg\phi$. On the other hand, if Φ is consistent with ϕ and consistent with $\neg\phi$, then ϕ is independent of Φ . For Hilbert one of the most important features of

³ The axiom groups are: axioms of incidence (group I: 1-7); axioms of order (group II: 1-5); axiom of parallels, or “Euclid’s Axiom” (group III); axioms of congruence (group IV: 1-6); and the axiom of continuity [*Stetigkeit*], or “Archimedes’ Axiom” (group V).

the axiomatic method is that it makes it possible to present and explicate, through the provability relation, the informal proof relations that obtain or fail to obtain between various kinds of mathematical beliefs or propositions. In other words, it makes it possible to distinguish between those propositions that are independent of one another, and so those whose justification is not inferential, and those propositions that are derived from the basic ones. This section presents an analysis of the consistency and independence proofs found in chapter V of (1899), Hilbert’s axiomatic analysis of Desargues’ Theorem, and shows that implicit in Hilbert’s use of the simplicity condition in (1899) is found the nascent concept of an extension, both a proto-model-theoretic concept of domain extension and a proto-proof-theoretic concept of a conservative extension. Our analysis not only provides a clear meaning to Hilbert’s demand that the axioms be “mutually independent” and “simple” but also shows how two of the metamathematical virtues that are often taken to be merely aesthetic play an objective role in the context of Hilbert’s axiomatic method.

2.2 Hilbert (1899) and Desargues’ Theorem

Projective geometry owes its birth to Renaissance efforts to represent three dimensional scenes on two dimensional canvases, and Desargues’ Theorem owes its origin to Desargues’ efforts to characterize the invariant properties of two dimensional representations when three dimensional scenes are spied from a different perspective. Indeed, when his work was rediscovered in the 19th Century, it was found that his theorem characterized the central concept of *perspectivity*. Desargues’ Theorem asserts:

(DT) if two triangles are perspective from a point, then they are perspective from a line.⁴

In chapter V of (1899) Hilbert employs the axiomatic method to study the provability relations between (DT) and the incidence axioms. Theorem 32

⁴ Triangles $\triangle ABC$ and $\triangle A'B'C'$ are in *perspective* from a line if the extensions of the three points of corresponding sides meet in collinear points. We consider only the planar version in what follows. Note its Π_2 form, and see footnote XX.

is a proof that (DT) “holds [*Gültigkeit*]” under the assumption that the planar incidence axioms (I: 1-2), the full order axioms (II), Hilbert’s axiom of parallels (III), and the full congruence axioms (IV) hold.⁵ Theorem 33 is a proof that when the above axioms are assumed—with the *exception* of SAS congruence (IV: 6)—(DT) is “unprovable [*Nichtbeweisbarkeit*].” Both proofs proceed by constructing interpretations of the terms that, under the relevant assumptions, either satisfy or fail to satisfy (DT). Thus, theorem 32 demonstrates that (DT) is consistent with the plane axioms of HG, whereas theorem 33 demonstrates that the negation of (DT) is consistent with those axioms; jointly, then, that it is independent of incidence geometry.

From the rediscovery of projective geometry in the 19th Century emerged the projective dualities, pairs of theorems one of which is obtained from the other by applying a scheme of uniform substitution—the duality theorems—for geometric terms: *point* is substitutable by *line*; *lie on* by *pass through*; *collinear* by *concurrent*; and *intersect* by *joins*. In projective plane geometry, for example, by applying the duality theorems to:

(I: 1) for any two points there is a unique line passing through both,

one obtains:

(I:1)’ for any two distinct lines, there is an unique point incident with both.

In order to obtain (I: 1)’ by the duality scheme, to a Euclidean 2-space one must add *ideal* points, points that are at an “infinite distance” in order for there to be a point at which any two lines meet, in particular for the special case when the two distinct lines in question are parallel.⁶ Extending the real affine plane by adding ideal points in the interpretation makes it possible to prove theorems of projective plane geometry. But though (I: 1)’ holds in the extension, (I: 1) does *not* hold, since two distinct ideal points in the projective plane do not uniquely determine a straight line, but a set

⁵ See Hallett and Majer (2004), p. 420 for a list of changes between the two editions.

⁶ The interpretation of the terms occurring in the incidence axioms that is extended by ideal points has the property that any two lines meet (the elliptic parallel property), distinct from the usual affine plane in which one works (which has the Euclidean parallel property), where some lines, i.e., parallels, do not meet.

of parallels. In order for both (I: 1) and (I: 1)' to hold, one must extend again by adding the *ideal* line, so that every Euclidean straight line meets the ideal line in exactly one point. The interpretation that contains ideal points and the ideal line is the projective completion of the affine plane.

Consider Hilbert's proof of theorem 32. On the assumption that (I: 1-2), (II), (III), and (IV) hold, Hilbert obtains the synthetic, "intuitive" interpretation of the non-logical terms. Working in the informal metatheory, the Euclidean plane, the goal is to extend the interpretation such that the extension satisfies (DT). Hilbert maps line segments into algebraic numbers, thereby mapping the synthetic interpretation into its algebraic coordinate interpretation, assigning points in the plane to pairs of coordinates for line segments (x, y) in Ω ; and lines in the Euclidean plane to ordered sets of ratios of three segments $(u : v : w)$ in Ω such that either $v \neq 0$ or $u \neq 0$, defined as the set of all pairs $((x, y), (u : v : w))$ for which:

$$ux + vy + w = 0,$$

the condition for incidence of point and line. He extends the interpretation to an algebraic coordinatization of the real projective plane. Reinterpreting "point" as an ordered triple of three segments (x, y, z) in Ω that are not all zero, and reinterpreting "plane" as an ordered set of ratios of four segments $(u : v : w : r)$ in Ω , where u, v , and w are not all zero, and "line" is reinterpreted as the set of all (x, y, z) with common solutions to two linear equations, or as the "intersection" of two "planes." Hence:

$$ux + vy + wz + r = 0$$

is the reinterpretation of "incidence" of point and plane. Hilbert's coordinatization of the projective plane satisfies (DT), where ideal points and the ideal line are associated with any irrational algebraic numbers that satisfy the second equation. Hence, the only remaining step is to demonstrate that (DT) is satisfied in the algebraic coordinatization of the affine plane. Hilbert observes that by setting $z = 0$ in the second equation, the first equation is recovered, thereby completing the proof (see fig. 2).

First, note that Hilbert does not assume that the spatial incidence axioms (I: 3-7) hold as hypotheses. In the usual affine plane interpretation, it is not possible to pick points outside of that plane and construct another plane intersecting it. Second, note that what makes it possible to pick a point outside of the original plane is an enrichment of the interpretation, the projection of an ideal point not lying in the Euclidean plane, but in the projective plane. Since, by hypothesis, (IV: 6) holds, one then constructs a triangle in the projective plane that shares at least one of its vertices with a triangle in the Euclidean plane, and is congruent to the latter triangle. By adding the ideal line, one projects “infinitely far” the sides of the triangle lying in the new plane, and those of one of the triangles lying in the Euclidean plane, such that the lines meet in ideal points collinear on the ideal line, showing that (DT) is consistent with the axioms in force. By assuming that ideal points exist, one extends the domain and increases the semantic range of the interpretation, thereby making it possible to prove (DT) from the usual set of incidence axioms, here (I: 1-2) and (II)-(IV), and increasing the range of provability in the axiomatic theory. However, the addition of ideal points alone has the effect of invalidating (I:1), thereby decreasing the range of provability in the axiomatic theory. Hence, in order to recover the full range of provability for the set of axioms, one must assume that the ideal line exists in the interpretation.⁷

Hilbert’s proof of (DT) in chapter five of (1899) illustrates the interplay between syntactic provability in the axiomatic theory and semantic extensions of the domain of the interpretation. Because of the presence of ideal points and the ideal line, the converse of (DT) holds, since one need only apply the duality scheme and uniformly substitute “point” for “line.” One preserves the validity of the set of axioms and obtains two proofs for the price of one. In section 24 of chapter five, Hilbert assumes that (DT) holds in conjunction with the planar incidence axioms, the order axioms, Euclid’s axiom, but absent the congruence axioms. By reinterpreting segments as numbers, Hilbert proves that the laws of arithmetic introduced in

⁷ In effect, to the metatheory one adds a set of propositions of Σ_2 form, in order to prove, in the axiomatic theory, a set of propositions of Π_2 form.

chapter three of (1899) are satisfied in an arithmetical interpretation. In this interpretation, commutative, associative, and distributive laws hold for addition, while for multiplication only the associative and distributive laws hold. Hence, the axioms in force implicitly define a “desarguean” plane geometry, which is then reinterpreted as what Hilbert baptizes a “desarguean number system,” a proper substructure of the complex field, namely the substructure in which commutativity fails. Hence, the consistency of the aforementioned axioms implicitly defines an interpretation, a concrete geometry that is then reinterpreted as a concrete substructure of the complex number field. When balanced such delicate extension and contraction procedures increase the interrelations between the basic and complex beliefs captured by an axiomatic theory, making it a presentation of those beliefs that is as efficient as it is powerful.

2.3 *Grundthatsache* and the Ideal Method

Hilbert’s distinction between basic beliefs, the *Grundthatsache*, and complex beliefs precedes but tracks his later distinction between “real” and “ideal” propositions made to carry out HP. In (1922), Hilbert calls the method of adding ideal propositions to real propositions in order to prove the consistency of extensions of the formal system the “ideal method.” Much of the literature on the consistency program describes Hilbert as an instrumentalist, that ideal propositions are “meaningless in themselves” in the sense that the only propositions that are meaningful are the real propositions, and that ideal propositions are meaningful only if they shorten deductions between the real propositions.⁸ In (1898c), for example, Hilbert writes that the use of ideal points and the ideal line “is entirely methodological” ((1898c), 167). One might think, then, that some evidence for this reading is to be found in Hilbert’s geometric period. But this misses the central point of Hilbert’s conception of the role of ideal elements. It is not, as he writes only lines later, a question of whether ideal elements exist, and hence, not a question about whether the propositions in which such terms occur are true indepen-

⁸ See, for example, Field (1980), Kitcher (1976), and Detlefsen (1986).

dently of the theory in which they are asserted.⁹ From Hilbert’s point of view, such questions are “completely idle,” for the addition of ideal elements makes it possible “to develop analytic geometry in its entire extent” (ibid.). In other words, by adding ideals one completes the real line, in the sense that to each real number there corresponds an ideal point. By extending the domain, one preserves provability in the axiomatic theory: the extension ensures that no theorem provable in analytic geometry is unprovable in synthetic geometry.

Consider the proof of theorem 33 in (1899), the proof that (DT) is *not* a consequence of (I: 1-2), (II)-(III), (IV: 1-5), and (V). The missing axiom is (IV: 6), the SAS congruence axiom.¹⁰ Hilbert’s goal is to prove that (DT) is unprovable without the aid of all of the congruence axioms, in particular, the axiom that binds angles to sides in triangle constructions. Hilbert begins the construction in the ordinary Cartesian plane. First one takes an arbitrary line in the plane as an axis and assigns to it an origin O . Hilbert fixes the axis to be the horizontal x -axis and introduces half-planes relative to it, the plane above the axis and the plane below the axis. He then constructs an ellipse with semi-axes at $[1, 0]$ and $[0, 1/2]$. Fixing another point on the x -axis at $[3/2, 0]$, Hilbert constructs circles from that point that intersect the ellipse in at most two real points. Then Hilbert reinterprets the Euclidean points as positions (x, y) on the Cartesian plane. So, Euclidean lines are preserved with one exception: if a Euclidean line intersects the ellipse in two Euclidean points, then the line segment between those two points is reinterpreted as the arc of a circle through $[3/2, 0]$, where the ellipse now functions “intuitively” like a convex fish-eyed lens for lines passing through it. Consider now the following: the line through O , the point $[3/5, 2/5]$, the x -axis, and the y -axis. In the ordinary Cartesian plane, two triangles with

⁹ Zach (2005), 19ff argues for this point with respect to Hilbert’s treatment of ideals during the later finitistic program. Also see, for a thorough exposition of the point vis-à-vis Hilbert’s geometric period, Hallett (1990), 239ff.

¹⁰ Hilbert takes SAS as an axiom, whereas Euclid attempts to prove it by a superposition argument that “lifts” the triangle out of the plane in which it lies and “moves” it on top of the given triangle. Euclid’s hidden assumptions include the spatial reasoning used in lifting the triangle, and the assumption that rigid bodies are invariant under motions, i.e., that they do not stretch, or bend, when in motion.

vertices on these three lines are perspective from a line. But here linear and angle structures are not preserved, so (DT) and SAS fail (see Fig. 3).

The proof of theorem 32 demonstrates that (DT) is consistent with the aforementioned subset of HG, and theorem 33 demonstrates that the negation of (DT) is consistent with the aforementioned subset of HG. Since the spatial axioms are necessary and sufficient conditions for its proof, (DT) has “spatial content,” the significance of which is that (DT) is a theorem of *plane* projective geometry ((1899), 70-1).¹¹ Moreover, its independence shows that any theorem provable with any of the spatial axioms is provable with (DT). Herein lies an important distinction. For Hilbert, consistency and independence are *inferential* conditions on the provability relation for a set of axioms, in the sense that if a set of axioms is inconsistent, then it is possible, *ex falso quodlibet*, to prove anything from them. Likewise, if an axiom is independent of another set of axioms, then it is not possible to prove that axiom (or its negation) from the set. On the other hand, domain extensions and the preservation of axiomatic provability are *explanatory* conditions, in the sense that if axiomatic provability is preserved come what may, the provability relation provides less of an explanation of those relations between the *Grundthatsachen* and the complex beliefs. On the other hand, the more elements by which the domain is extended tends to increase the explanatory power of the axioms, but also tends to decrease the extent to which the syntactic provability relation is preserved. Hence, though contemporary metamathematicians often assimilate them, in Hilbert’s geometric period, the concept of a conservative extension plays a distinctively different theoretical role than the concept of axiomatic consistency.

During his geometric period, Hilbert understands axiomatic mathematical theories in analogy with scientific theories. Showing that an axiomatic mathematical theory is consistent is intended to provide the mathematician with justification for believing that the axioms and their consequences are true. But his method of domain extensions to the semantic interpretation and the preservation of the provability relation has the role of providing the

¹¹ For discussion of its spatial content against the background of Hilbert’s remark, in the conclusion to (1899), about the “purity of method,” see Hallett (2008), 309-16.

mathematician with an explanation of the relationship between the basic beliefs codified by the axioms and the more complex beliefs that are, or ought to be, derivable. For Hilbert, the balance between extensions of the domain, by ideals, and the preservation of provability in the base theory, the *Grundthatsache*, does not necessarily permit the mathematician to infer that because extensions by ideal elements do not invalidate any one of the axioms in the base theory, the extended theory is dispensable, that he can merely work in, or be justified in believing, the simpler set of axioms as opposed to the more complex ones. Hence, it seems that instrumentalist interpretations of the geometric and later programs, insofar as they take the conservativity of the extension over the base theory to show that the extension is in principle dispensable, are too quick. In the next two sections, our goal is to disentangle Hilbert’s conceptions of completeness [*Vollständigkeit*] so that a better understanding of the range of possibilities emerges.

2.4 *Vollständigkeit*, Continuity, and Categoricity

In the last section we discussed domain extensions and the preservation of axiomatic provability and showed that unlike consistency these methods are explanatory rather than inferential. In this section we turn to Hilbert’s addition, in the first French edition (1900b) and the second German edition (1903), of the *Axiom der Vollständigkeit*. The literal translation of “vollständig” is ‘complete’, and that of “Vollständigkeit” is ‘completeness’. The *Axiom der Vollständigkeit* (VA) is absent from (1899). When it does appear in subsequent additions, it appears as (V: 2) alongside Archimedes’ Axiom (V: 1) as an axiom of continuity. Intuitively, in (1899) Archimedes’ Axiom ensures that to each point on the line there corresponds a real number. But still there may be real numbers that satisfy the axioms but that do not correspond to points on the line. In (1899) such points must be constructed “piecemeal,” by the addition of ideal elements or by the construction of limit points, additions that provide for the “local” completeness of the constructions that require them. But the addition of (VA) in (1900b) assuages that need, since it ensures that to each real number there exists a point, and so ensures “globally” that there are sufficiently many points on

the line that correspond to real numbers. In the next two sections, we argue that there are at least two senses in which Hilbert uses the concept of completeness. In this section, we isolate completeness in its “model-theoretic” sense. In the section after that, we isolate its “proof-theoretic” sense.

In order to be clear about the difference between the proto-model-theoretic and proto-proof-theoretic senses of “completeness,” here we’ll quote the French and German versions of (VA). In (1900b), the *Axiom d’Intégrité* asserts that to the “système des points, droites et plans,”

il est impossible d’adjoindre d’autres êtres de manière que le système ainsi généralisé forme une nouvelle géométrie où les axiomes des cinq groupes I-V soient tous vérifiés; en d’autres termes: les éléments de la Géométrie forment un système d’êtres qui, si l’on conserve tous les axiomes, n’est susceptible d’aucune extension. ((1900b), p. 123)

In (1903), axiom V: 2, the *Axiom der Vollständigkeit*, is a more or less direct translation of the French, and asserts:

[d]ie Elemente (Punkte, Geraden, Ebenen) der Geometrie bilden ein System von Dingen, welches bei Aufrechterhaltung sämtlicher genannten Axiome keiner Erweiterung mehr fähig ist, d.h.: zu dem System der Punkte, Geraden, Ebenen ist es nicht möglich, ein anderes System von Dingen hinzuzufügen [sic], so dass in dem durch Zusammensetzung entstehenden System sämtlich aufgeführten Axiome I-IV, V 1 erfüllt sind. ((1903), p. 16)

The most natural and straightforward translation of (1903) takes “Erweiterung” as ‘extension’, and “erfüllt” as ‘fulfilled’ or ‘satisfied’, yielding:

the elements (points, lines, planes) of geometry form a system of things that are capable of no extension [*Erweiterung*] while preserving all of the listed axioms; in other words, to the system of points, lines, and planes it is not possible to adjoin [*hinzufügen*] another system of things, so that in the resulting composite system [*Zusammensetzung entstehenden System*] the entire presentation of axioms I-IV, V: 1 are fulfilled [*erfüllt*].

Note that, at least on the surface, the axiom appears to refer to the intended interpretation, all other possible interpretations of the terms occurring in HG, and to the axioms themselves.

2.5 Conclusion

In sections (2.2) and (2.3) we studied domain extension and contraction procedures in the context of Hilbert’s axiomatic analysis of Desargues’ Theorem, and showed that such procedures play an explanatory role in addition to an inferential role. Section (2.4) isolated the model-theoretic constraints imposed on HG by Hilbert’s *Vollständigkeitsaxiom* by contrasting his approach with the Dedekind-Cantor set-theoretic approaches, and showed that Hilbert’s analysis makes explicit hidden higher-order categoricity that the Dedekind-Cantor approach leave implicit. The above discussion shows an essential relation to modern metamathematical concepts employed in HP during the 1920’s and beyond, such as conservativity, consistency, categoricity, and completeness. We now connect the geometric program to the emergence of HP in the 1920s.

In (1929) Hilbert discusses four unsolved problems for finitistic proof theory, his 1920’s proposal for the foundations of mathematics. Problem one discusses consistency proofs and proof-theoretic reductions for different types of mathematical theories. Problem two discusses consistency proofs and reductions for formal systems of functionals of higher types. Problem three discusses the completeness problem for formal systems of arithmetic and analysis. Problem four discusses the completeness problem for formal systems of logic. Hilbert’s discussion of problem one makes it clear that reductions are to be carried out by conservativity proofs. But Hilbert does not specify whether he has in mind model-theoretic conservation, in which it is shown that given two models one can always be contracted to another, or proof-theoretic reduction, in which it is shown that given two formal theories every formula provable in one is provable in the other. In his discussion of the third problem, Hilbert points out that “the usual argument with which one shows that any two realizations of the axiom system for number theory, respectively, of analysis, must be isomorphic, do not satisfy the demands

of finitistic rigor” ((1929), 231). He claims that the problem is “transform the usual proof of isomorphism finitistically” (ibid.) in order to show that if a statement in formal arithmetic is consistent then it is provable. In his debate with Frege, Hilbert claims that consistency of a statement implies its intuitive truth. Hence, the latter problem implies the need to show that if a statement is arithmetically true, then it is provable in formal arithmetic.

The introduction of ideal points and the ideal line increases both the efficiency and power of the axiomatic theory, and helps explain the relationship between the basic beliefs of an intuitive theory and its more complex beliefs. Likewise, we might think that a conservativity proof does not show that one can “dispense with” the theory being shown conservative (cf. Field, Detlefsen) but that it is intended to be an explanation of the relationship between the two theories. An essential component of this explanation is to elucidate hidden higher-order concepts that are, or might be, implicit in mathematical practice. But Hilbert’s geometric work is also a study in how different formal systems present the mathematical and metamathematical phenomena that they are intended to present. Because it is a useful means of reconsidering the contemporary philosophical and mathematical role of HP, we now turn to Hilbert’s Program in the 1920’s and some of the post-Gödelian in HP.

3 Finitism and Hilbert’s 1920s Program

By the early 1920s, Hilbert had settled on the outlines of HP, both epistemic and mathematical. In short, the idea is that the mathematician adopts what he called the finite standpoint [*finit Standpunkt*], and from there constructs a consistency proof, however that is understood, of the axioms of classical mathematics. But before we continue further there are some important distinctions to make. It’s useful to view (HP) as having two distinct tiers. At the first, pre-theoretic, tier lies the distinction between finitary and infinitary mathematics. This is the “school arithmetic” tier in which we make judgments involving equalities and inequalities between numerals and learn to construct and decompose these numerals intuitively by iteration. It’s at this level that the finitary class of judgments can be called “intuitive” or “finitely

verifiable.” Because this is a pre-theoretic and unanalyzed tier, the distinction between finitary and infinitary arithmetic remains vague. But at the second, theoretic tier, one views arithmetic as a formal theory regimented in a formal language. Those judgments that were intuitively understood to be finitary at the pre-theoretic level form a class of sentences that Hilbert calls “real,” to be contrasted with those sentences that are infinitary at the pre-theoretic level and that now form a class called “ideal.” However, it is not simply that one views arithmetic in this setting as a formal theory. Rather, one regiments a set of arithmetical judgments into a formal language in order to provide a rigorous *analysis* of the judgments, processes, and distinctions that does not affect practice at the pre-theoretic tier. It’s an important, but often overlooked point. For although many have understood Hilbert’s epistemological view to be vaguely instrumentalist, instead Hilbert understood his program as one that neither presupposed nor entailed any philosophical supplementation. For one of the goals of the consistency proof was to eradicate a kind of philosophical skepticism about the reliability of arithmetic that had produced theory-laden “solutions” to the classical antinomies such as Frege’s platonism and Brouwer’s constructivism.

So, at the pre-theoretic tier it is possible, but unnecessary, to adopt whatever position one wants towards arithmetic — platonist, constructivist, whatever. Hilbert’s idea was to leave the pre-theoretic tier of arithmetic alone, to view it in the most natural way when we approach it from the standpoint of someone practicing arithmetic. For Hilbert this means that there is a natural point of view when we operate with numbers in which the iterative manipulation of finite numbers and finite sequences (and, for that matter, the intuitive use of induction in proofs of simple arithmetical theorems) is bedrock and basic. It is this tier and this point of view towards arithmetic and mathematics generally that informs the epistemological position that Hilbert dubbed finitism. However, it’s at the second tier that Hilbert’s Program is to be understood as a philosophical description of mathematical activity whose goal is to explain how the finitist at the first tier can regard the practice of infinitary mathematics as consistent with his rational standards. It’s at this second tier where one deploys real sentences

in order to explain how ideal sentences are not at odds with them (read: not inconsistent with), just in case such sentences “save” or preserve our most natural way of approaching arithmetic (and mathematics more generally). One might, per the platonist or nominalist, nonetheless cite our epistemic situation towards the objects of arithmetic and insist on the literal truth or falsity of ideal sentences, but to Hilbert it was unnecessary baggage. From the standpoint of a practicing mathematician, the only constraints were the “natural” point of view arising from the manipulation of finite sequences of finite numbers.

In (1922) Hilbert stated these constraints in detail in a well-known paragraph, one that he would repeat almost verbatim from the early 1920s onward. From the finitistic standpoint, the precondition for logical inference is that “something must be already given in representation [*Vorstellung*], certain extra-logical discrete objects” that are “intuitive,” “immediate,” and prior to discursive thought. Moreover, he continues, if logical inference is:

certain [*sicher*], these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their concatenation (like the objects themselves) must be given to us immediately and intuitively, as something that cannot be reduced to something else. ((1922), 202)

Finitists represent in intuition “concrete” sequences of strokes of the form:

$$|||| = ||||$$

by representing a single stroke (or a sequence of strokes) and, by concatenations and decompositions of strokes over given representations, one generates, or “constructs,” other representations. Hilbert is not much more precise about the exact structure of finitistic reasoning, but at times he characterizes its epistemic pedigree as resulting from the fact that these strokes and the elementary concatenation and decomposition operations are “representable in intuition,” or “presupposed by all thought,” and at other times that they constitute a distinct “source of *a priori* knowledge.” Indeed it’s likely that his belief that the consistency proof would be successful

led him to believe that, once complete, whatever philosophical imprecisions existed in the account would be rendered inconsequential.

Owing to Hilbert’s opacity assessments of the success or failure of these epistemic constraints vary widely in the literature. Kreisel (1983) argues that the defining feature of finitistic reasoning is that it is “completely surveyable,” but his analysis has not been endorsed. Kitcher (1976) argues that the central concept is “representability in intuition,” but rejects finitism as a viable epistemology of mathematics since infinite sequences of numerals are not representable in intuition. Like Kitcher, Tait (1980) rejects “representability in intuition” as epistemologically significant, and concludes that there is no basis for Hilbert’s distinction between finitistic and non-finitistic reasoning. He argues that the distinguishing feature of finitistic reasoning is that it is “presupposed by thought” in that “it forms a nontrivial part of mathematics presupposed by any treatment of number” (Tait (1980), 540). However, in (1926) Hilbert provides two perspicuous examples of claims that are justified from the finitistic standpoint. The first, $3 > 2$, Hilbert writes, “serves to communicate the fact that the sign 3 (that is, |||) extends beyond the sign 2 (that is, ||), or that the latter sign is a proper segment of the former” ((1926), 377). If understood as an indication of synonymy, Hilbert’s “that is” is a mistake. He means that the proposition expressed by the claim “ $3 > 2$ ” serves to communicate the fact constructed in intuition that one representation of strokes is a proper segment of the other representation of strokes. It is the representation that serves to justify the content of the claim using Arabic numerals and the sign for inequality.

The second example he provides is the claim that given a prime number p , there exists some prime q such that:

$$p + 1 \leq q \leq p! + 1.$$

Hilbert claims that it serves to abbreviate a disjunction of finite length:

$$q = p + 1 \text{ or } q = p + 2 \text{ or } \dots \text{ or } q = p! + 1,$$

where q is a prime. However, the assertion that there is some prime q , such that $q > p$ is *not* finitistically justified, since “no matter how harmless

the transition appears to be, there is a leap into the transfinite when this partial proposition [...] is stated as an independent assertion” ((1926), 378). First note that for the finitist the content of a belief expressed by a claim involving relations between constants has logical structure, and hence, a truth-value. But the representations that serve to justify her claims are constructed in intuition, and so are “extra-logical,” or “non-conceptual.” Hence, finitistic claims are truth-apt, but finitistic representations are not, since the latter cannot serve as the propositional content of a belief because they have no propositional structure. Of course, this does not entail that they have no epistemic structure. Second note the category of beliefs the finitist characterizes as *a priori* justified. On one hand, the first example is an inequality between constants. We might expect that similar beliefs, e.g., that $5 > 4$, are also *a priori* justified, since there seems to be no principled distinction between the manner in which the two are justified. On the other hand, the second example is a claim involving an existential quantifier. But there is a numerical bound on the quantifier, so the finitist claims that it is equivalent to a finite disjunction of claims involving only numerals. Such examples suggest a natural category distinction between closed term equations, inequalities, and bounded quantificational claims, and all other claims, in particular claims that involve potentially infinite sequences or completed infinite sets. This is how, in the literature on finitism and HP, there is near universal agreement that the category of claims including closed term equations, inequalities, and bounded quantifications is unequivocally finitistically justifiable. But what *else* is finitistically justified.

Hilbert writes that besides claims such as the above “we encountered finitary propositions of problematic character, for example, those that were not decomposable,” into partial propositions possessing a finite upper bound that are then analyzable into finite truth-functional expressions ((1926), 380). He calls these claims “problematic” and illustrates with the claim that, if $\mathfrak{a}$ and $\mathfrak{b}$ are numerals, then:

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a},$$

Hilbert writes that problematic claims are “from the finitist point of view *incapable of being negated*” ((1926), 378). Hilbert argues that finitists ought

to regard problematic claims as hypothetical judgments, since unlike closed term equations, they are incapable of being true or false. Suppose, for example, that the commutativity claim in section above is capable of being negated. Then, like closed term equations, if it is true, it must be satisfied by every numeral, and if it is false, then there must exist a counterexample. But that follows from unrestricted excluded middle when the domain of the property instantiated by the principle is the set of natural numbers (or rather, the set of stroke-sequences isomorphic to the natural numbers), the content of which is *not* finitistically justified. Hence, if the finitist is to pass beyond the finitistic base, he must be in a position to explain how problematic claims admit of finitistic justification. How? Bernays' procedure permits finitists to justify claims about finite numbers such as:

$$10^{10^{10}} = 10^{10^{10}}$$

not by a single finitistic representation *simpliciter*, but by a kind of reflection and recombination of representations. If so, then it seems possible for the finitist to obtain a justification for more complex (but still closed term) claims from less complex claims such as:

$$1 = 1,$$

by forming the belief that if k is a numeral, then $k - 1$ applications of modus ponens are sufficient to find her way back to a claim about finite numbers that are unequivocally justified.

In general, from the finitistic standpoint it is possible to construct a numeral k (or a sequence of strokes, or more generally, a sequence of elements taken from a finite alphabet) from an arbitrarily given numeral j by iterating the successor operation $k - j$ times. No matter how we spell out the conception of finitism (e.g., Hilbert's "representability in intuition"), it is clear that it must provide us with a means of justifying at least the successor function and the constant function, since we need both a starting point and a means of obtaining larger numerals from that starting point. More precisely, for an arbitrarily given numeral x_k , finitists can read:

$$(1) \ 0(x_k) = 0$$

$$(2) \ S(x_k) = x_k + 1,$$

$$(3) \ U_i^k(x_1, \dots, x_k) = x_i,$$

with $1 \leq i \leq k$, as descriptions, respectively, for constructing the initial finite ordinal from a given finite ordinal, constructing the successor of a finite ordinal from a given finite ordinal, and (re)constructing the i th finite ordinal from a list of k numerals. These “descriptions” are listed in order of dependence. One begins from the initial ordinal by (1), constructs the k th ordinal by (2), and by (3) ignores all the ordinals in the list of length k except the i th ordinal. Second note that the order of dependence implies that if the finitist is justified in believing (1), then since (2) is constructed by appealing to (1), (2) is also finitistically justified. Likewise, one’s justification for believing that (1) and (2) are finitistic implies that (3) is finitistic, since (3) is constructed by appealing only to (1) and (2). As finitists, we begin with evidence or justification for (1), and in virtue of it construct (2). Then, we construct (3) in virtue of our evidence or justification for (1) and (2). Hence, it is possible for us to construct operations of greater complexity from operations of lesser complexity, when and only when we are “given” the operations of lesser complexity.

However, the problem is not with cases in which “small” integers are calculated from “small” integers, in particular in cases where the only operation under consideration is an iteration of successor. Consider, then, a function such as exponentiation. Suppose that the ϕ_{ith} function is x^y and that the finitist knows that x^y (for $x = 10$ and $y = 10$) is finitistic. Suppose further that the ϕ_{i+1st} function, the next iteration of that function for those values, is $10^{10^{10}}$, and that it is not finitistic. One might be able to construct (or represent to oneself) “small” integers such as 10^{10} but fail to be able to construct much larger ones such as $10^{10^{10}}$ due to the difference in size. Though we considered this case briefly above, a similar case is raised by Bernays’ in his elucidation of Hilbert’s finitism, so let’s look at Bernays’ response to this problem in more detail. If one knows that 10^{10} is finitistic, then one iterates, as above the process used to obtain 10^{10} tenfold in order to obtain $10^{10^{10}}$. That the process of construction by which the latter number

is produced from the former number cannot fail to be finitistic can be seen in the following discussion by Bernays. He writes that finitist methodology:

is not concerned with such thresholds for the possibility of the realization [of constructing a number like $10^{10^{10}}$ from 10^{10}]. For these thresholds are accidental from the point of view of the formal consideration. Formal abstraction, as it were, does not find an earlier position for a delimitation in principle than the difference between finite and infinite. (Bernays (1930), 249-50)

In this quote, by “formal abstraction,” Bernays has in mind something like the following mathematical argument. Suppose that there is an upper bound, or threshold, on the finite numbers, above which no ordinal construction reaches. Let z be that number. Since z is a finite number, it can be represented by a finite set Z . By Cantor’s Theorem, Z has a finite power set 2^Z such that the cardinality of the latter is strictly larger than that of the former. Since 2^Z is larger than Z , Z is the largest and not the largest set. We reach a contradiction (where the sets considered are clearly decidable), and it follows that no such bound exists. But it does not follow that the set of functions are unbounded. Bernays’ claim is that whatever the mathematical upper bounds are, they can be adduced neither by empirical considerations, nor by considering the limits of our powers of representation, nor by arbitrary conceptual stipulations (e.g., Tait (1981), Kitcher (1976)), but only by mathematical means. If we know that one function is constructible by one means and is finitistic, then if the “next” function is constructed in a sufficiently similar manner to the previous function, it too must be finitistic. That is, if the evidential bases for the i th and $i + 1$ st stages are relatively uniform, the successor stage can be accepted.

In effect, we are asking about the kinds of considerations that we are able to adduce for limits on the functions that can be constructed from the finitistic point of view. Bernays’ claim is that no empirical reasons can count for those limits since otherwise it would follow that iterations of values for a constructed and known mathematical function can be bounded above by non-mathematical, philosophical means. In such a case, the iteration of

values of a function might be justified as finitistic to a sufficiently high degree for knowledge for some values but for some larger value fail to be finitistic. It is not clear upon what grounds, other than the processes being sufficiently *distinct* or the finitist being insufficiently reliable, that one might declare one function that produces a finite number to be finitistic and another function that produces a larger finite number to be non-finitistic. On the other hand, Bernays does not tell us much more in (1930) about the kinds of mathematical considerations finitists are permitted to (or forbidden from) adduce in their constructions of functions. For example, if the i th stage of construction yields each primitive recursive function, and the finitist knows that each primitive recursive functions has been constructed (but not all of them together), and knows that each is finitistic, then if the ϕ_{i+1st} function is the function that outputs each primitive recursive function, a general but not primitive recursive function, is it finitistic or does it fail to be so? It was this question, in fact, and not the more narrowly construed question about the consistency of mathematics, that took up the bulk of work in Hilbert's Program in the 1920s. Below we look at the proof to help understand.

3.1 HP & the Ackermann Function

For example, Ackermann's dissertation (1924) is one of the earliest major contribution to Hilbert's finitist proof theory and Hilbert's Program following Hilbert and Bernays's development in 1922 and 1923 of the ϵ -calculus, their formalism for the analysis of arithmetic and analysis. He was a student of Hilbert's, and only four years after his dissertation, with Hilbert Ackermann co-authored *Grundzüge der theoretischen Logik*. In his dissertation Ackermann develops in succession four types of basic calculi and presents consistency proofs for each one. That Ackermann's consistency proofs were considered, historically, to be finitistic, there can be no doubt. In a letter of recommendation for a fellowship awarded in 1925 Hilbert reports:

Ackermann has shown in the most general case that the use of the words “all” and “there is”, of the “tertium non datur”, is free from contradiction. The proof uses exclusively primitive

and finite inference methods. Everything is demonstrated, as it were, directly on the mathematical formalism.¹²

For our purposes the important consistency proof is the one that takes the consistency proof given for an elementary calculus of free-variables, quantifier-free axioms, and schemata for generating primitive recursive definitions and extends it to an elementary calculus with free-function-variables. Ackermann calls the first calculus stage III (in keeping with the Hilbert School’s “stepwise” and “simultaneous” development of logic and mathematics alongside one another), and calls the second calculus stage IV. Hilbert had already given a consistency proof for a version of stage III (essentially PRA) in lectures from 1921-22. In (1924) Ackermann follows Hilbert’s lead from those lectures. First, one must assume, for *reductio*, that there is a linear derivation of $0 \neq 0$, then transform that derivation into one of tree-form, eliminate variables, and reduce function terms to numerals. At the end of the process one obtains a sequence of correct formulae, where “correct” is syntactically defined and decidable. One then concludes that there can be no derivations of $0 \neq 0$. But since no function variables are admitted at stage III, the verification of the argument is straightforward. But for stage IV it must be shown that every function is reducible to a numeral on the basis of the recursion equations when functions are permitted to take functions as arguments. Herein lies the crux of the problem.

Ackermann notes the problem with the following example.¹³ Let

$$\phi_m(2, \mathfrak{g}(m))$$

be a functional of higher type such that ϕ is defined by:

$$\phi_m(0, f(m)) = f(1) + f(2)$$

$$\phi_m(n + 1, f(m)) = \phi_m(n, f(m)) + f(n) \times f(n + 1)$$

¹² Hilbert-Nachlass, Niedersächsische Staats- und Universitätsbibliothek, Göttingen. Cod. Ms. Hilbert 458, sheet 6, undated. Quoted in Zach (2000).

¹³ In what follows we follow Ackermann’s own presentation from (1924) but change some notation, most notably the notation for ranks, indices, and orderings. Ackermann uses natural numbers in his proof, but we use ordinal notations for perspicuity.

Note that here $\mathfrak{g}(m)$ is a term for a function that occurs within a function. Hence, it is not possible to replace the variable b occurring as its argument before evaluating the entire function. Reduce the functional to the following form by applying the recursion equations:

$$\begin{aligned}
\phi_m(2, \mathfrak{g}(m)) &= \phi_m(1 + 1, \mathfrak{g}(m)) \\
&= \phi_m(1, \mathfrak{g}(m)) + \mathfrak{g}(1) \times \mathfrak{g}(1 + 1) \\
&= \phi_m(0 + 1, \mathfrak{g}(m)) + \mathfrak{g}(1) \times \mathfrak{g}(1 + 1) \\
&= \phi_m(0, \mathfrak{g}(m)) + \mathfrak{g}(0) \times \mathfrak{g}(0 + 1) + \mathfrak{g}(1) \times \mathfrak{g}(1 + 1) \\
&= \mathfrak{g}(1) + \mathfrak{g}(2) + \mathfrak{g}(0) \times \mathfrak{g}(1) + \mathfrak{g}(1) \times \mathfrak{g}(2) + \mathfrak{g}(1) + \mathfrak{g}(2)
\end{aligned}$$

Since $\mathfrak{g}(m)$ is a term for a function, the function ϕ itself might be in its domain. For example, $\mathfrak{g}(m)$ might denote the function $\phi_j(m, \gamma(j))$. But if so, then $\phi_m(2, \mathfrak{g}(m))$ reduces to:

$$\begin{aligned}
&\phi_j(1, \gamma(j)) + \phi_j(2, \gamma(j)) + \phi_j(0, \gamma(j)) \\
&\quad \times \\
&\quad \phi_j(1, \gamma(j)) + \phi_j(1, \gamma(j)) \\
&\quad \times \\
&\quad \phi_j(2, \gamma(j)) + \phi_j(1, \gamma(j)) + \phi_j(2, \gamma(j)).
\end{aligned}$$

Hence, it is possible to reduce a functional of higher type to a form in which it occurs as argument. On the other hand, if one reduces a primitive recursive term $\psi(\mathfrak{m})$ to a numeral, where ψ and $\mathfrak{m}$ are given, using the recursion equations for the function, then, unlike the reduction of the above term, ψ will not be contained in the final form of the reduced term.

In other words, we have not shown that the reduction terminates in a numeral. Rather we've shown that it is possible to reduce the given function to another function if functions take themselves as argument. But not only must we show that it terminates. Also we must provide a justification of the associated method that shows that finitists are entitled to employ it. Ack-

ermann proceeds in two steps as follows. First, he assumes that innermost subterms are reduced first, and then assigns indices to terms to show that each reduction reduces the given index. Then he proves that the reduction of functionals of higher type terminates in a numeral. On that basis he concludes that the calculus is, in fact, consistent. We'll proceed with the proof as follows. Definitions precede the statement of the theorem. The inequality $\phi < \psi$ denotes that ϕ occurs before ψ in the order of definition.

Definition 4.0 (subordination): if t is a term containing ϕ and ψ , then ψ is *subordinate* to ϕ , abbreviated $sub(\psi, \phi)$, just in case:

$$t = t'(\dots \phi(\dots \psi(\dots b \dots) \dots) \dots),$$

where b occurs bound in ψ .

Definition 4.1 (rank): if $\psi > \phi$, then the *rank* of the function symbol with respect to its occurrence in t , abbreviated $rk(t, \phi)$, is defined as:

$$rk(t, \phi) = \begin{cases} 1 & \text{if no } \psi \text{ is } sub(\psi, \phi) \\ \max\{rk(t, \psi) \mid sub(\psi, \phi)\} + 1 & \text{otherwise.} \end{cases}$$

Definition 4.2 (order): if $\mathfrak{s}$ is a subterm of $\mathfrak{t}$ and $\psi_0 < \dots < \psi_n$, then the *order* of the subterm $\mathfrak{s}$, abbreviated $o(\mathfrak{s})$, is defined by:

$$o(\mathfrak{s}) = \omega^n \cdot rk(\mathfrak{s}, \psi_n) + \dots + \omega \cdot rk(\mathfrak{s}, \psi_1) + 1 \cdot rk(\mathfrak{s}, \psi_0)$$

Definition 4.3 (degree): if $\mathfrak{s}$ is the set of all distinct subterms of t with an order o where the order is not a numeral, then the *degree* of o in t , abbreviated $d(t, o)$, is that set's cardinality, defined by:

$$d(t, o) = |\mathfrak{s}|$$

Definition 4.4 (index): if o ranges over all orders of subterms of t , and $\vec{o}$ is the ordinal corresponding to o , then the *index* i of t , abbreviated $i(t)$,

is the sequence of degrees ordered identically to the orders, defined by:

$$i(t) = \sum_o \omega^{\vec{o}} \cdot d(t, o).$$

Note that if $\psi_0, \dots, \psi_n$ is a sequence of primitive recursive functions (defined as in (3.3)) then the *order* of $\mathfrak{s}$, $o(\mathfrak{s})$, corresponds to an ordinal α such that $\alpha < \omega^\omega$, and the *index* of $\mathfrak{s}$ corresponds to an ordinal $\beta < \omega^{\omega^\omega}$.

We are now in a position to prove Ackermann's theorem for formal languages containing free function variables.

Theorem 4.0: Every higher type functional reduces to a numeral — or terminates in a numeral — on the basis of its recursion equations.

Proof: Let $\mathfrak{t}$ be a term not containing function symbols for transfinite recursive functions, and let $\mathfrak{s}$ be an arbitrary constant subterm occurring innermost in the term $\mathfrak{t}$ such that $\mathfrak{s}$ has the form:

$$\mathfrak{s} = \phi_{\vec{b}}(\mathfrak{a}_1, \dots, \mathfrak{a}_n, \mathfrak{v}_1, \dots, \mathfrak{v}_m)$$

where the $\mathfrak{v}_i$'s are terms with at least one variable bound by ϕ and don't contain constant subterms. There are two cases to consider.

Case One: $m = 0$, that is, $\mathfrak{s}$ contains no bound variables. If $m = 0$, then since $\psi_{k-1} > \phi_{\vec{b}}$ does not occur in $\mathfrak{t}$ and there's no occurrence of $\psi_k > \phi$ subordinate to ϕ in $\mathfrak{t}$, it follows that:

$$\begin{aligned} o(\mathfrak{s}) &= < rk(\mathfrak{s}, \psi_k), \dots, rk(\mathfrak{s}, \psi_1), rk(\mathfrak{s}, \psi_0) > \\ &= \omega^k \cdot 1 + \dots + 0 + 0 \\ &= \omega^k \end{aligned}$$

Using the recursion equations to reduce $\mathfrak{s}$ to a term $\mathfrak{s}'$ it is clear that only function symbols with indices i lower than the index for ψ_k occur. Hence, the highest exponent in the order of $\mathfrak{s}'$ is less than k . Hence, $o(\mathfrak{s}') < o(\mathfrak{s})$. Since, by assumption, no variable which is bound occurs in $\mathfrak{s}$, no variable bound in $\mathfrak{t}$ occurs in $\mathfrak{s}'$. Hence, reducing $\mathfrak{s}$ to $\mathfrak{s}'$ results

in no new occurrences of function symbols subordinate to any other function symbols. It follows, when $\mathfrak{t}'$ is the result of the reduction of $\mathfrak{t}$ in this way, that the number of subterms of orders greater than $\mathfrak{s}$ in $\mathfrak{t}'$ remains constant, but that the number of subterms of orders exactly equal to $o(\mathfrak{s})$ in $\mathfrak{t}'$ decreases by 1. Hence:

$$i(\mathfrak{t}') < i(\mathfrak{t}).$$

Case Two: $m > 0$, that is, $\mathfrak{s}$ contains bound variables. Without loss of generality let $\mathfrak{s}$ be of the form:

$$\mathfrak{s} = \phi_b(\mathfrak{z}, \mathfrak{g}(b)),$$

that is, that $\mathfrak{s}$ contains one argument with an arbitrary constant term $\mathfrak{z}$ and one argument with an arbitrary functional $\mathfrak{g}$. All function symbols occurring in $\mathfrak{g}(b)$ are subordinate to ϕ , since otherwise it would contain a constant subterm and this case would collapse to the evaluation of a (first-order) primitive recursive function. Since function symbols occurring in $\mathfrak{g}(b)$ are subordinate to ϕ , it follows that:

$$rk(\mathfrak{t}, \mathfrak{g}(b)) < rk(\mathfrak{t}, \mathfrak{s}).$$

Using the recursion equations to reduce $\mathfrak{s}$ results in a term $\mathfrak{s}'$ that does not contain the symbol ϕ . Note that if one substitutes an arbitrary term $\mathfrak{b}$ for the variable b in $\mathfrak{g}(b)$, then because none of the occurrences of function symbols in $\mathfrak{b}$ contain bound variables whose scope is outside of $\mathfrak{b}$, so that no bound variables are subordinate to any function symbols in $\mathfrak{g}(b)$, it follows that:

$$(*) \quad o(\mathfrak{g}(\mathfrak{b})) = \max\{o(\mathfrak{g}(b)), o(\mathfrak{b})\}.$$

We must show that reducing $\mathfrak{s}$ to $\mathfrak{s}'$ in $\mathfrak{t}$ decreases $\mathfrak{t}$'s index. The proof now proceeds by induction on the term $\mathfrak{z}$.

Let the following be the recursion equations for ϕ :

$$\phi_b(0, f(b)) = \mathbf{a}_b(f(b))$$

$$\phi_b(a + 1, f(b)) = \mathbf{b}_b(\phi_c(a, f(c)), a, f(b)).$$

Base Case: If $\mathbf{s} = \phi_b(0, \mathbf{g}(b))$, then $\mathbf{s}' = \mathbf{a}_b(\mathbf{g}(b))$. We must argue that $\mathbf{s}'$ has the same rank as $\mathbf{g}(b)$ with respect to ϕ . Since $\mathbf{a}$ does not contain ϕ , when $f(b)$ is an argument, it's replaced by $\mathbf{g}(d)$ such that the variable d is not in the domain of ϕ . That shows that $rk(\mathbf{t}, \mathbf{s}') \leq rk(\mathbf{t}, \mathbf{g}(b))$. In the other direction, with a term $\mathbf{g}(\mathbf{d})$ such that $\mathbf{d}$ does not contain $\mathbf{g}$, by $(*)$ and the fact that $\mathbf{d}$ does not contain ϕ , it follows that $rk(\mathbf{t}, \mathbf{g}(\mathbf{d})) = rk(\mathbf{t}, \mathbf{g}(b))$. If $\mathbf{d}$ does contain an occurrence of a nested $\mathbf{g}$, then by the induction hypothesis and the argument in case one, $rk(\mathbf{t}, \mathbf{d}) = rk(\mathbf{t}, \mathbf{g}(b))$.¹⁴ Hence, by $(*)$, $rk(\mathbf{t}, \mathbf{s}') \geq rk(\mathbf{t}, \mathbf{g}(b))$. Hence:

$$rk(\mathbf{t}, \mathbf{s}') = rk(\mathbf{t}, \mathbf{g}(b)).$$

Induction: If $\mathbf{s} = \phi_b(\mathbf{z} + 1, \mathbf{g}(b))$, then $\mathbf{s}' = \mathbf{b}_b(\phi_c(\mathbf{z}, \mathbf{g}(c)), \mathbf{z}, \mathbf{g}(b))$. Further applications of the recursion equations replace the first argument, $\phi_c(\mathbf{z}, \mathbf{g}(c))$, by a sequence of other terms $\mathbf{u}'_0, \dots, \mathbf{u}'_k$ such that for each $0 \leq i \leq k$, by k applications of the induction hypothesis, we have:

$$rk(\mathbf{t}, \mathbf{u}'_i) \leq rk(\mathbf{t}, \mathbf{g}(b)).$$

Hence, in this case it too follows that:

$$rk(\mathbf{t}, \mathbf{s}) \leq rk(\mathbf{t}, \mathbf{g}(b)).$$

Hence, if $\mathbf{t}'$ results from eliminating ϕ from $\mathbf{t}$ by recursion, then the argument shows that the rank of $\mathbf{t}'$ decreases by at least 1 and the recursion does not increase the rank of any subterms ψ_j for $j > i$.

¹⁴ Note that because we are dealing with *nested* recursions, the base case includes a (series of) sub-induction(s), and the inductive case includes a series of sub-inductions, one for each evaluation of the function using the recursion equations.

It is important to note that since the index of $\mathfrak{s}$ is an ordinal ω^{ω^k} for some k , a formal argument for the above claim requires transfinite induction on each ordinal $\alpha < \omega^{\omega^\omega}$ in order to show that in a finite number of steps one obtains a functional that contains no function symbols — that is, a numeral. But Ackermann describes the theorem above as showing only that the computation procedure defined by 2-fold nested recursions such as the function given by the sequence $\psi_0, \dots, \psi_n$ terminates. He writes that:

[t]o each functional corresponds a transfinite ordinal number as its rank, and the theorem, that a constant functional is reduced to a numeral after finitely many operations, corresponds to [the other theorem] that if one descends from a transfinite ordinal number to ever smaller ordinal numbers, one must reach zero after a finite number of steps. (Ackermann (1924), 13-4)

And he sees nothing in it that violates the finitistic standpoint:

there is naturally no mention of transfinite sets or ordinal numbers in our metamathematical investigations. It is, however, interesting that the theorem about transfinite ordinals can be formulated so that there is nothing transfinite about it. (Ibid.)

Ackermann proceeds to argue that the finite sequences of numbers in the finitistic version of the theorem are well-ordered. Importantly, however, he does not argue for the well-orderedness of the entire class of sequences of sequences (i.e., for each α -recursive function such that $\alpha < \omega^{\omega^\omega}$), but only for that subset of the sequences pertaining to the 2-fold nested recursion that bears his name (i.e., for $\alpha < \omega^2$). We reproduce the argument below.

Ackermann asks us to consider:

a transfinite ordinal number less than $\omega \cdot \omega$. Each such ordinal number can be written in the form $\omega \cdot (\mathfrak{n} + \mathfrak{m})$, where $\mathfrak{n}$ and $\mathfrak{m}$ are finite numbers. Hence such an ordinal can also be characterized by a pair of finite numbers $(\mathfrak{n}, \mathfrak{m})$, where the order of these numbers is of course significant. (Ackermann (1924), 14)

Once we have a characterization of the ordinals up to ω^2 in terms of a pair of finite numbers, Ackermann characterizes the reduction of the pair to $(0, 0)$ in terms of the operations performed on them. He continues:

[t]o the descent in the series of ordinals corresponds the following operation on the number pair $(\mathfrak{n}, \mathfrak{m})$. Either the first number $\mathfrak{n}$ remains the same, then the number $\mathfrak{m}$ is replaced by a smaller number $\mathfrak{m}'$. Or if the first number $\mathfrak{n}$ is made smaller, then I can put an arbitrary number in the second position, which can also be larger than $\mathfrak{m}$. It is clear that one has to reach the number pair $(0, 0)$ after finitely many steps. For after at most $\mathfrak{m} + 1$ steps I reach a number pair, where the first number is smaller than $\mathfrak{n}$. Let $(\mathfrak{n}', \mathfrak{m}')$ be that pair. After at most $\mathfrak{m}' + 1$ steps I reach a number pair in which the first number is again smaller than $\mathfrak{n}'$, etc. After finitely many steps one reaches the number pair $(0, 0)$ in this fashion, which corresponds to the ordinal number 0. In this form, the mentioned theorem contains nothing transfinite whatsoever. Only considerations which are acceptable in metamathematics are used. (Ibid.)

Ackermann's argument appeals to our ability to record the order of the applications of the recursion equations over a pair of finite numbers. As one considers one or the other number contained in the pair, one need only apply the equations finitely many times over that pair in order to reach the point at which the application of the equations terminates. Note that an argument that shows that a given primitive recursive function terminates need only apply the equations finitely many times over a single arbitrary finite number. Hence, the only difference between Ackermann's argument and one that shows that a primitive recursive function terminates is that his argument proceeds over *pairs* of arbitrary finite numbers and not a single arbitrary finite number. Ackermann appears to take that similarity to be evidence for the claim that 2-fold nested recursion is finitistic.

If we accept the similarity as evidence for the claim, then we obtain a justification for extending the finitistic standpoint from its characterization

in a formalism in which individual primitive recursive functions are definable to a formalism in which all the primitive recursive functions are definable by 2-fold nested recursion. In terms of an ordinal characterization, we obtain a justification for passing from each ordinal $\alpha < \omega^\omega$ to each ordinal $\alpha < \omega^{\omega^2}$. But is it possible to extend — with justification of equal degree — the claim that each ordinal $\alpha < \omega^{\omega^2}$ is finitistic to each ordinal $\alpha < \omega^{\omega^\omega}$? Ackermann claims that it is:

[t]he same holds true if one does not use pairs but triples, quadruples, etc. This idea is not only used in the following proof that the reduction of functionals terminates, but will also be used again and again later on, especially in the finiteness proof at the end of the work. (Ibid.)

There are three points to consider. First, Ackermann’s description of the process serves as a set of instructions for reducing recursion on ω^2 to primitive recursion when substitution in the parameters is permitted. Since that process can be reduced to primitive recursion, it shares the same epistemic justification as primitive recursion. Second, the schemata of quantifier-free induction and recursion on ω^k for each k are derivable in PRA instance by instance. If the functions definable in PRA are finitistic (and are by most accounts in the literature), and since the 2-fold general recursive function bearing Ackermann’s name is a description of that set of functions, then it must be finitistic. Third, and on the other hand, if it is possible to further iterate Ackermann’s justification, then his description ought to justify induction on each ordinal up to ω^ω . For “the same holds true” comment seems to claim that the justification for the construction process at one stage, since it warrants acceptance at that stage, warrants it at the next stage. But note that Ackermann purports to provide a finitistic proof of the induction principle on ω^ω , and a proof must appeal to induction on ordinals beyond ω^ω . That is, Ackermann’s proof involves a tacit appeal to induction on ordinals whose status as finitistic is not justified by his description. Hence, Ackermann’s description provides a justification for recursion on ω^2 but fails to iterate further and hence fails to provide a justification for recursion on each ordinal up to ω^ω .

3.2 HP & Hilbert and Bernays' *Grundlagen*

However, written in the 1920s and finally published in the 1930s (after Gödel's seminal results), sections (2) and (7) of Hilbert and Bernays' *Grundlagen der Mathematik* (GMI) and (GMII) discuss extensions of the finitist standpoint to functions beyond the primitive recursive functions. Most of the discussion of (GMI) and (GMII) in the secondary literature on the scope of finitism has focused on section (7) (see, e.g., Zach (1998), Tait (2000)), since therein is the most extended discussion of the Ackermann function and the enumeration function for the primitive recursive functions. Tait concludes from their discussion that the authors fail to prove that general but non-primitive recursive functions are finitistic. However, in section (7) the authors refer back to section (2), in which they introduce what they call *finitist number theory* (FNT), a “contentual” non-formal theory of primitive arithmetic the authors consider to be finitistic, in contrast to what they call *recursive number theory* (RNT), a formal theory of arithmetic one version of which includes a quantifier-free induction scheme (and is equivalent to PRA). In contrast to Ackermann's purported proof that each ordinal up to $\omega^{\omega^{\omega}}$ is finitistic, in (GMI) Hilbert and Bernays offer a means weaker than proof — what they call a “verification” — that they suggest justifies the extension of the finitistic standpoint from the primitive recursive functions to general recursive but non-primitive recursive functions. In this section we analyze section (2) of (GMI), then section (7) of (GMI).

Section (2) of (GMI) is an extended discussion of (FNT) with respect to its admissible operations. Some operations, claim Hilbert and Bernays, are *independently* justified as finitistic, while other operations are *dependently* justified as finitistic. The operations that follow from the basic operation of constructing a numeral by iterating over given sequences in representation are independent, while those that stand in need of some sort of justification — such as the principle of induction, or definitions by recursion — are dependent ((GMI), 23). Recall, from chapter two, that an arbitrary finite numeral consists in a representation of finite sequences constructed by iterating over a given representation such as $|$. Hence, these representations are of the form $|\cdots|$ and constitute (FNT)'s domain ((GMI), 21). It is now pos-

sible to state the language for which its “communication signs” are defined. Let $\mathcal{L}_{FNT}$ consist (at least partially) in the following:

- *constants*: ‘2’ abbreviates ‘11’, ‘3’ abbreviates ‘111’, etc., and ‘11’, ‘111’, etc., refer to the represented stroke-sequence numerals ||, |||, etc.;
- *variables*: ‘a’, ‘b’, etc., refer to arbitrary stroke-sequences of numerals;
- *operations*: ‘+’ refers to the operation of addition over numerals, ‘.’ refers to the operation of multiplication over numerals, etc.;
- *constructions*: ‘ ϕ ’, ‘ ψ ’, etc., refer to defined (function) constructions;
- *relations*: ‘=’ and ‘ $\neq$ ’ refer to numerical identity and difference.

Hilbert and Bernays are explicit about the fact that the language is *not* closed. For as one develops (FNT) new communication signs (for abbreviations and the introduction of functions) might be needed (ibid.). In other words, if it is possible to finitistically derive, based on previously derived or primitively given facts or rules, other facts or rules, then the derived facts or rules may be taken as new facts or rules in (FNT). The authors provide a simple example. Suppose the stroke-sequence denoted by a numeral **a** “matches” the sequence denoted by a numeral **b**. Then one of the two sequences is an initial segment of the other or they are identical. Hilbert and Bernays claim that trichotomy follows from the justification above, and that similar considerations justify transitivity ((GMI), 22). Hence, it is legitimate to extend $\mathcal{L}_{FNT}$ by the signs ‘<’ and ‘>’ and to extend (FNT) by rules for inequalities between numerals (interpreted in (FNT)’s domain).

After introducing addition (whose justification rests on our ability to append stroke-sequences to stroke-sequences) into (FNT), Hilbert and Bernays claim that associativity for addition is independently justified based on considerations involving stroke-sequences. But the authors then argue that commutativity for addition is not independently justified because it requires an appeal to the “proof method of complete induction [*die Beweismethode der vollständigen Induktion*]” ((GMI), 23). Hilbert and Bernays are discussing

mathematical induction, or more precisely, quantifier-free induction over the numerals, and induction over the numerals, they argue, requires a finitistic justification in order to be introduced into (FNT). Since its character is reflected in their subsequent discussion of recursive functions, let's briefly look at their justification for it. Hilbert and Bernays ask us to consider a predicate $\mathfrak{P}$ that one wants to conclude holds of an arbitrary numeral $\mathfrak{a}$. In order to conclude that $\mathfrak{P}$ holds for arbitrary $\mathfrak{a}$, one must derive:

- (i) $\mathfrak{P}(1)$;
- (ii) $\mathfrak{P}(\mathfrak{n}) \rightarrow \mathfrak{P}(\mathfrak{n} + 1)$.

Hilbert and Bernays argue that if $\mathfrak{a}$ is an arbitrary numeral, then it has a construction procedure that begins with a representation of a sequence of strokes to which the basic iteration and concatenation operations are applied finitely-many times. In order to derive (i) and (ii), one verifies that $\mathfrak{P}$ holds for $\mathfrak{a}$ in tandem with the means by which $\mathfrak{a}$ is constructed. Hence, (i) and (ii) are derived co-temporally with $\mathfrak{a}$'s construction (ibid.).¹⁵ According to the authors, the above is a sufficient justification for introducing quantifier-free induction into (FNT). Moreover, it is now possible to prove, using the introduced principle, the commutativity of addition.¹⁶

Theorem 4.1: For any arbitrary numerals $\mathfrak{a}, \mathfrak{b}$: $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$.

Proof: The proof proceeds by induction on numerals $\mathfrak{a}$ and $\mathfrak{b}$.

Base Case: Since the term $1 + 1$ is syntactically equivalent to the term $1 + 1$ it follows that $1 + 1 = 1 + 1$ immediately.

¹⁵ Hilbert (1931a) argues that the above construction is the “intuitive-contentual” justification of induction. One difference between (1931a) and (1934) is that in (1934) the authors are clear that the *proofs* of (i) and (ii), the *verification* that $\mathfrak{P}$ holds of $\mathfrak{a}$, and the *construction* of the numeral $\mathfrak{a}$ occur co-temporally.

¹⁶ The proof below depends, in three steps, on a lemma that establishes that addition is commutative for the case in which $\mathfrak{a} + 1 = 1 + \mathfrak{a}$. Hilbert and Bernays do not derive commutativity for addition in (1934) at all, but mention that it is derivable, and discuss the mentioned lemma. On the other hand, in (1921/22), Hilbert derives the theorem 4.1, but proceeds by reducing numerals on either side of the identity side to obtain the identity $0 = 0$. Hence, Hilbert's proof in (1921/22) demonstrates that the procedure of commuting over addition terminates, and hence does not depend on the lemma.

Induction: Let $\mathbf{n}$ and $\mathbf{m}$ be arbitrary numerals and assume:

$$\mathbf{n} + \mathbf{m} = \mathbf{m} + \mathbf{n}.$$

Syntactic identity between terms gives us immediately:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = \mathbf{n} + (1 + \mathbf{m} + 1).$$

By the associativity of addition, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = \mathbf{n} + (1 + \mathbf{m}) + 1.$$

By commutativity for arbitrary numeral $\mathbf{a} + 1$, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = \mathbf{n} + (\mathbf{m} + 1) + 1.$$

By the associativity of addition, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = (\mathbf{n} + \mathbf{m}) + 1 + 1.$$

By the inductive hypothesis and substitution, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = (\mathbf{m} + \mathbf{n}) + 1 + 1.$$

By the associativity of addition, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = \mathbf{m} + (\mathbf{n} + 1) + 1.$$

By commutativity for arbitrary numeral $\mathbf{a} + 1$, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = \mathbf{m} + (1 + \mathbf{n}) + 1.$$

By the associativity of addition, it follows that:

$$\mathbf{n} + (1 + \mathbf{m} + 1) = (\mathbf{m} + 1) + (\mathbf{n} + 1).$$

By commutativity for arbitrary numeral $\mathfrak{a} + 1$, it follows that:

$$\mathfrak{n} + (1 + \mathfrak{m} + 1) = (1 + \mathfrak{m}) + (1 + \mathfrak{n}).$$

By the associativity of addition, it follows that:

$$(\mathfrak{n} + 1) + (\mathfrak{m} + 1) = (1 + \mathfrak{m}) + (1 + \mathfrak{n}).$$

Since the claim holds for the base case and the inductive case, then by the quantifier-free induction principle it follows that commutativity of addition for arbitrary $\mathfrak{a}, \mathfrak{b}$ is derivable in (FNT). $\square$

We derived theorem 4.1 in order to provide an example of how, in (FNT), it is possible to justify new rules and facts based on previously justified rules and facts. By the argument from our section (3), the degree to which theorem 4.1 is justified as finitistic depends upon the degree to which induction is finitistic, in the sense that if it is known that induction is finitistic then the degree of confidence that proofs based on induction are finitistic must be high enough to warrant knowledge that induction is. Following the discussion of commutativity and some properties of multiplication, Hilbert and Bernays discuss the method of introducing a function sign via definition by recursion. Hilbert and Bernays write:

[a] new function-sign, such as ϕ , is introduced and its definition is presented through two equations, which in the simplest cases have the form [*im einfachsten Falle die Form haben*]:

$$\begin{aligned}\phi(1) &= \mathfrak{a} \\ \phi(\mathfrak{n} + 1) &= \psi(\phi(\mathfrak{n}), \mathfrak{n}).\end{aligned}$$

Here $\mathfrak{a}$ is a numeral and ψ is a previously known function constructed according to previously known rules so that it is possible to calculate $\psi(\mathfrak{b}, \mathfrak{c})$ given numerals $\mathfrak{b}, \mathfrak{c}$ such that its calculation results in a numeral as value. ((GMI), 25-6)

Below, we look at Hilbert's introduction of the Omega-Rule, in 1931, as a further step in the consideration of work on (HP).

3.3 HP & the Omega-Rule

Hilbert introduces the ω -rule in (1931a) with the claim that proofs for two theorems are the most important task for the finitist. These theorems are:

1. If a statement can be shown to be consistent, then it is also provable; and furthermore,
2. If for some statement $\mathfrak{S}$ the consistency with the axioms of number theory can be established, then it is impossible to also prove for $\bar{\mathfrak{S}}$ [i.e., $\neg\mathfrak{S}$] the consistency with those axioms. ((1931a), 1154)

Hilbert claims that for certain simple cases of formal theories he has successfully found finitistic proofs for (1) and (2). He argues that success had been made possible for these cases by the introduction of a new rule. Let's call this Hilbert's Rule (HR) and, following Hilbert, state it as follows:

(HR): Let $\mathfrak{A}(\mathfrak{z})$ be a formula in the language of (FNT), with $\mathfrak{z}$ arbitrary. Then, if $\mathfrak{A}(\mathfrak{z})$ is proved as correct for each numerical instance, then it is legitimate to introduce $\forall x A(x)$ as an axiom.

(HR) should not be viewed as a standard inference rule stated in a formal theory.¹⁷ Instead, one ought to understand it as permitting the introduction of a universally quantified formula into a formal system if one has provided a finitistic justification for each of its instances. Hilbert claims that he obtained proofs for (1) and (2) based on “adding to the already given rules of inference (substitution and inference schema) the following equally finite new rule” (ibid.). From this it is clear that he believes that the use of this rule in finitistic consistency proofs is finitistically justified.

In Hilbert (1931a) there is not much provided in the way of a defense of the status of (HR). Hilbert reminds us that “the statement $\forall x A(x)$ extends far wider than the formula $\mathfrak{A}(\mathfrak{z})$, where $\mathfrak{z}$ is an arbitrary given numeral. For in the former case not merely a numeral, but any expression of our formalism having a numerical character can be substituted for x in $A(x)$; moreover,

¹⁷ With this claim we follow the interpretation of the ω -rule given by Sieg (2010). This point becomes relevant below where we discuss interpretations of (HR) in the literature.

the negation can be formed in accordance with the logical calculus” (ibid.). In addition, he provides a proof of theorems (1) and (2) for $\mathfrak{S}$ restricted to Π_1 formulae and proves (2) for $\mathfrak{S}$ restricted to Σ_1 for a system F of elementary arithmetic plus the new “rule” (HR).¹⁸ Moreover, Sieg (2010) writes that we ought to consider the putative finitistic character of the proofs in (1931a) in contrast to methods that, in a talk given in Bologna two years earlier, Hilbert claims are non-finitistic. Sieg writes that in the Bologna talk Hilbert points out that completeness for arithmetic and analysis is usually assumed to hold and that these claims are grounded in their (second-order) categoricity. But Hilbert then claims that arguments for categoricity do not meet the demands of finitist rigor. Hence, “[Hilbert] suggests, as a next step, transforming the standard categoricity proof for number theory into a finitist argument that would establish [theorems (1) and (2)]” (Sieg (2010), 6). In (1931) Hilbert first proves that $F + HR$ is (relatively) consistent, taking for granted that Ackermann and von Neumann have proved the consistency of first-order arithmetic. He argues that if the extended system is inconsistent, then there is a “proof-figure” Φ obtained from the addition of (HR) whose end-formula is a contradiction. Since Φ consists in initial formulae (axioms) or formulae obtained from initial formulae by inference rules, its end-formula must be obtained from one of those. Since F is consistent, the end-formula must be obtained from (HR) and be of the form $\forall xA(x)$. By instantiation, it follows that $\mathfrak{A}(\mathfrak{a})$ for a determinate numeral. But by the antecedent of (HR) $\mathfrak{A}(\mathfrak{z})$ has been shown to be correct for arbitrary $\mathfrak{z}$. Hence, contradiction. Hence, there is no such proof-figure. Hence, $F + HR$ is consistent. Hilbert turns to the proof of (1) and (2) for “certain simple” cases:

Theorem 4.3: For each Π_1 formula, $F + HR$ satisfies (1) and (2).
 $\text{---}0;11?''''''''i$

Proof: Let $\mathfrak{S}$ be of the form $\forall xA(x)$, where x is the only bound variable, and let $\forall xA(x)$ be consistent with $F + HR$. If the formula $\mathfrak{A}(\mathfrak{a})$ obtained from $\forall xA(x)$ by substitution is not correct and hence not prov-

¹⁸ With this claim we follow the interpretation of the formal system in (1931a) given by Sieg (2010) and Ignjatović (1994). Again it is important to our discussion below of (HR).

able, then the formula $\neg\mathfrak{A}(\mathfrak{a})$ is provable and hence correct. Hence $\exists x\neg A(x)$ is provable. But then, contrary to hypothesis, $\forall xA(x)$ is inconsistent with $F + HR$. Hence, each Π_1 formula of the form $\forall xA(x)$ containing no variable other than x is provable, and (1) follows. Since $\mathfrak{S}$ is Π_1 , theorem (2) follows from theorem (1) ((1931a), 1155). $\square$

It is important to note at this point that, whatever $F + HR$ may be, it is *not* a standard formal system in Hilbert’s (earlier) more traditional sense. Nor is it like (FNT) or (RNT) as understood in Hilbert and Bernays (1934). Rather, due to the fact that (HR) is formulated such that its antecedent itself requires finitistic justification that, when given, permits (by its consequent) the introduction of universally quantified statements as axioms (initial formulae) in a formal system, we might think of it as a semi-formal system.¹⁹ In (1931a) Hilbert notes that Ackermann’s and von Neumann’s consistency proofs for arithmetic (both for systems equivalent to (PA)) demonstrate that “the transfinite modes of inference — in particular the mode of inference of *tertium non datur* — can be seen to be admissible in the domain of elementary number theory” ((1931a), 1154). Moreover, Hilbert describes the axioms as axioms for his proof theory. Hence, Hilbert takes $F + HR$ to be a (semi-formal) enrichment of the metamathematical proof-theory and that his task (through the proof of consistency) is to show it is consistent in order to apply it to an object theory.

If so, then the question arises as to the precise interpretation of Hilbert’s extension. Consider that Hilbert’s proof of theorems (1) and (2) for Π_1 formulae appears to be a proof that $F + HR$ is Π_1 -complete, in the sense that every true Π_1 -formula possesses a proof in $F + HR$. There are (at least) four plausible interpretations of (HR) under which it can be made more precise. Either (i) (HR) specifies the set of all Π_1 formulae whose matrices are Π_0 -formulae ϕ such that some finitistic theory F proves that $\phi(\mathfrak{n})$ is provable for each $\mathfrak{n}$. Or (ii) (HR) is interpreted as specifying the set of Π_1 formulae whose matrices are Π_0 -formulae ϕ such that for each $\mathfrak{n}$ some

¹⁹ Feferman (1986) describes the system in Hilbert (1931a) as a “semi-formal” system. Here we follow that description. Though we must keep in mind in the following that it raises some interpretive objections. For a contrast see Niebergall and Schirn (2001).

finitistic theory F proves $\phi(\mathbf{n})$. More precisely:

- (i) $(\text{HR})_1 := \{ \forall x \phi(x) : \phi \in \Pi_0 \text{ and } F \vdash \forall x \text{Pr}_F(\ulcorner \phi(x) \urcorner) \};$
- (ii) $(\text{HR})_2 := \{ \forall x \phi(x) : \phi \in \Pi_0 \text{ and } F \vdash \phi(n) \text{ for each } n \}.$

From (i) and (ii) one obtains (i'), (ii'), (i''), and (ii'') by observing that either one takes only a *single* application of (i') $(\text{HR})_1$ or (ii') $(\text{HR})_2$ in the extension, or one takes *all* applications of (i'') $(\text{HR})_1$ or (ii'') $(\text{HR})_2$ in the extension. Note that we do not have a precise mathematical understanding of either $(\text{HR})_1$ or $(\text{HR})_2$ insofar as we have not specified to what F amounts. On the one hand, if F is PRA (or one of its conservative extensions), then $F + (\text{HR})_1$ is an axiomatizable subtheory of (PA), since (PA) proves uniform reflection for PRA, and (i') is provably equivalent to (i'').²⁰ It follows that $F + (\text{HR})_1$ is not Π_1 -complete as Hilbert's proof seems to demonstrate. On the other hand, if again F is PRA, then $F + (\text{HR})_2$, if it is closed under the addition of $(\text{HR})_2$, then (ii') is provably equivalent to (ii''). But then the extension $F + (\text{HR})_2$ is proof-theoretically equivalent to $(QF - IA) + T(\phi)$, where $T(\phi)$ is a formal truth-definition for formulae ϕ such that ϕ are Σ_2 . Since a set is recursively enumerable if and only if it is Σ_1 , it follows that $F + (\text{HR})_2$ is *not* recursively enumerable. But that seems to clash with the finite standpoint.²¹ Hence, if we take F to be (PRA), then the interpretations of (HR) (i') and (i'') pose an historical obstacle to Hilbert's textual claim, while (ii') and (ii'') seem to pose a conceptual obstacle for finitists.

A different situation arises if we interpret F as (PA). Under that assumption $F + (\text{HR})_2$ for both (ii') and (ii'') are provably equivalent, and since that $F + (\text{HR})_2$ is provably equivalent to $(PA) + T(\phi)$ where $T(\phi)$ is

²⁰ Although it may seem anachronistic to use PRA here, it ought to suffice to note that it is possible to replace (PRA) with (QF-IA) or $I\Delta_0$, both of which are proof-theoretic conservative extensions of PRA containing the usual arithmetic axioms plus first-order logic. The definition of $F + \text{HR}$ as $(QF - IA) + (\text{HR})_1$ above is presented in a slightly different but equivalent form (and then defended) in Ignjatović (1994).

²¹ Detlefsen (1979), 310, denies that the metamathematical framework needs to be recursively enumerable. We come back to his claim below. Note that if we adopt the strategy of interpreting F as PRA and (HR) as $(\text{HR})_2$, then $F + (\text{HR})_2$ proves the consistency of theories as strong as PA and even Zermelo-Fraenkel (ZF), if PA and ZF are consistent.

a formal truth-definition for formulae ϕ such that ϕ are Π_1 , $F + (HR)_2$ is Π_1 -complete just as Hilbert appears to demonstrate. In fact, Hilbert seems to claim that, pace Ackermann's and von Neumann's consistency proofs for (PA), it is legitimate to extend (PRA) — what we might take as the base — in the step-by-step way outlined by his program all the way to (PA) and then extend (PA) with (HR). However, although this interpretation agrees with Hilbert's claim to have demonstrated what appears to be the Π_1 -completeness of the semi-formal system, just because of this it is, again, not recursively enumerable. Hence, on this interpretation of $F + (HR)$ we have a genuine trade-off. On the one hand, since $F + (HR)$ is Π_1 -complete, and since the consistency formula for F is an instance of a Π_1 formula, it is possible to prove in $F + (HR)$ that F is consistent. On the other hand, since $F + (HR)$ is not recursively enumerable, the argument for F 's consistency cannot be formalized in F . Hence, we obtain traction on one of HP's central goals, but no longer have traction on to what the metamathematics amounts. Sieg has found evidence that suggests that Hilbert was aware of Gödel's results and that, in part, Hilbert's (1931a) is a response to the perceived difficulties (Sieg (2010), 8). Hence, it seems reasonable to think that some working finitists concluded that such arguments count as finitistic, despite the fact that they might not be formalizable.

But one of Hilbert's students and a central figure in work on (HP), Gerhard Gentzen, recognized the deeper difficulty in (HR). In Hilbert (1931a), (HR) contains an unanalyzed concept — that of a predicate being verified as “correct” for each numeral n — and that is of course one of the main hurdles in the analysis of its status. In (1936) Gentzen proves the consistency of (PA) by replacing the concept of correctness (Gentzen refers to it as “contentual correctness” [*inhaltlicher Richtigkeitsbegriff*]) with what he calls the “stability of a reduction rule” for derivations that occur in the formal theory to be analyzed. Each derivation is assigned an ordinal number that measures its complexity, and Gentzen shows that each reduction step decreases the complexity of the derivation. Once this is shown, Gentzen proves that for each reduction step, the ordinal number of a given derivation decreases. Hence, by iterating reduction steps on a given derivation, one

arrives at its end-sequent in finitely many steps. Once Gentzen has shown that only finitely-many steps are required for all derivations with ordinal numbers less than some ordinal α , it follows, by his proof for of this fact for transfinite induction, that this property holds for α itself and hence holds for *all* derivations in (PA) of arbitrary ordinal number. Gentzen claims that the universal quantifier in question must be taken as finitistic. He writes that “in each case we are dealing with a totality [the set of PA-derivations] for which a constructive rule for generating all elements is given” (Gentzen (1936), 560). Since he replaces Hilbert’s informally-defined concept of correctness with his own precisely defined concept of stability, Gentzen’s proof of the consistency of (PA) replaces a finitistic concept with a well-defined syntactic concept. Hence, unlike Hilbert’s proof using (HR), Gentzen’s need not appeal to informal finitistic concepts.²²

In fact, Gentzen identifies precisely the problem in Hilbert’s proof. On a Hilbert-style interpretation of (HR) in which the concept of correctness is replaced by a truth-definition, Gentzen writes that one might:

proceeding from these considerations, develop a purely formal consistency proof for this part of number theory. But such a proof would have little value, for in the proof itself one would have to use transfinite statements and the accompanying modes of inference that one wants to “ground” by the proof. So the proof would not be a proper reduction, but rather a confirmation of the finitist character of the formalized rules of inference. But one must already be clear in advance what counts as finitist (in order then to be able to carry out the consistency proof itself with finitist means of proof). (Gentzen, (1936), 529)

His charge is twofold. First, Gentzen objects to its circularity. One must use the formal system whose consistency is at issue in the consistency proof.

²² Here we do not attempt to answer the question whether Gentzen’s proof that (PA) is consistent using (PA) plus transfinite induction on ordinals up to ϵ_0 is finitistically justified. Gentzen claimed that it is. But our main focus here is his criticism of (HR). In (GMII) Hilbert and Bernays recognize Gentzen’s objection below and make remarks suggesting that his “purely syntactic” proof overcomes the objection ((1939), 390ff).

Second, Gentzen’s deeper point is that by using (HR) one is committed to a truth-definition for the very formulae whose truth is at issue in the consistency proof. Hence, Gentzen’s objection is neither that the addition of (HR) produces a set that is not recursively enumerable nor that the addition of (HR) permits methods beyond those recognizable as finitist. Both objections beg the question. For the metamathematics might not necessarily be recursively enumerable from an epistemic point of view, and we must first understand what is recognizable as finitist in order to claim that some method is not recognizable as such. Gentzen’s objection is fatal to (HR) because he shows that *no* non-circular methods justify it.²³

4 Conclusion

In the 1920s Hilbert’s School, which included mathematicians such as Ackermann, Bernays, Gentzen, Gödel, von Neumann, and many others, worked on problems far beyond a narrow focus on consistency. To be sure, much of the work was focused on obtaining a consistency proof for classical mathematics. But in addition, the bulk of work in HP during the 1920s was devoted to understanding finite sequences, quantification over sets, the then-nascent concept of recursion, both general and primitive, as well as deepening the mathematics behind algebraic reductions, the methods for independence proofs, and exploring the computational and proof-theoretic scope of a given fragment of mathematics; research that very broadly, can and should be categorized as research into the finitist standpoint. Consistency was an important, but not the only, problem for researchers in HP.

In fact by 1931, when Gödel published his results that would come to be known as (G1) and (G2), HP had opened up an enormously rich body of research that would serve as the starting point for almost every branch of mathematical logic, and many branches of mathematics as well, that came after Gödel’s (G1) and (G2). Even in (1929) Hilbert discusses four unsolved problems for finitistic proof theory. Problem one discusses consistency proofs

²³ Sieg writes that Hilbert’s (HR), “not clearly respecting the line between syntactic and semantic considerations, has been made blindingly clear by Gentzen” (Sieg (2010), 38).

and proof-theoretic reductions for different types of mathematical theories. Problem two discusses consistency proofs and reductions for formal systems of functionals of higher types. Problem three discusses the completeness problem for formal systems of arithmetic and analysis. Problem four discusses the completeness problem for formal systems of logic. Hilbert's discussion of problem one makes it clear that reductions are to be carried out by conservativity proofs. But Hilbert does not specify whether he has in mind model-theoretic conservation, in which it is shown that given two models one can always be contracted to another, or proof-theoretic reduction, in which it is shown that given two formal theories every formula provable in one is provable in the other. In his discussion of the third problem, Hilbert points out that "the usual argument with which one shows that any two realizations of the axiom system for number theory, respectively, of analysis, must be isomorphic, do not satisfy the demands of finitistic rigor" ((1929), 231). He claims that the problem is to "transform the usual proof of isomorphism finitistically" (ibid.) in order to show that if a statement in formal arithmetic is consistent then it is provable. This basic idea is what is known as a conservativity proof, carried out after the 1920s programmatically by, most notably, Feferman and Kreisel.

Hilbert's proposal for conservativity proofs in relation to his analysis of ideal points and the ideal line during his geometric period suggests that conservativity proofs appear to play an epistemic role in HP, too. The introduction of ideal points and the ideal line increases both the efficiency and power of the axiomatic theory, and helps explain the relationship between the basic stuff of an intuitive theory and its more complex sentences. But to show that a theory T' is conservative over T does not show that one can "dispense with" the theory being shown conservative, as is common to argue after Quine. Instead, pace the work in HP in the 1920s, such proofs *explain* the relationship between the two theories. In any case, work in proof theory after the 1930s bears many of these ideas out, and speaks to the rich body owing to the origin and execution of Hilbert's Program in the 1920's.

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